Form factors of pion and kaon

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In addition to the vector meson poles in the form factors of charged pions and kaons additional intrinsic form factors of pions and kaons are found. A detailed study of the form factors of pions and kaons in both timelike and spacelike regions is presented. Theory agrees well with data up to $q\sim 1.4$ GeV. In this study there is no adjustable parameter.

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The form factor is a very important physical quantity in understanding the internal structure of hadrons. Vector meson dominance (VMD) [1] is successful in studying electromagnetic interactions and the vector part of the weak interactions of mesons. According to VMD the pion form factor is determined by a ρ-meson pole [2]. Generally speaking, the ρ pole fits the data well. However, a detailed study shows that there is a deviation between the experimental data of the pion form factor and the ρ pole. The electromagnetic radius of charged pion has been determined to be [3]

$$\langle r^2\rangle_\pi^{exp} = (0.439 \pm 0.03) \quad \mathrm{fm}^2, \tag{1}$$

whereas the result obtained from the ρ-pole is

$$\langle r^2\rangle_\pi^{\rho\ pole} = 0.391 \quad \mathrm{fm}^2. \tag{2}$$

In Ref. [4] a comparison between the measurements and the ρ pole [2] of the pion form factor is presented. Figure 6.3 of Ref. [4] shows that the ρ pole [2] decreases a little bit faster in the timelike region and slower in the spacelike region. The experiments show that, in addition to the ρ pole, maybe an additional form factor is needed.

Based on 't Hooft's arguments that in large N_C limit QCD is equivalent to a meson theory [5] an effective chiral theory of large N_C QCD of mesons has been proposed [6]. It has been shown that the tree diagrams of mesons are at leading orders and the loop diagrams of mesons are at higher orders in large N_C expansion. In the chiral limit $m_q\to 0$, the Lagrangian of this theory in the case of two flavors takes the form

$$\mathcal{L} = \bar{\psi}(x)[i\gamma\cdot\partial + \gamma\cdot v + \gamma\cdot a\gamma_5 - mu(x)]\psi(x) + \frac{1}{2}m_0^2(\rho_i^\mu\rho_{\mu i} + \omega^\mu\omega_\mu + a_i^\mu a_{\mu i} + f^\mu f_\mu), \tag{3}$$

where $a_\mu = \tau_i a_\mu^i + f_\mu$, $v_\mu = \tau_i\rho_\mu^i + \omega_\mu$, and $u = \exp\{i\gamma_5(\tau_i\pi_i + \eta)\}$; m is a parameter; u can be written as

$$u = \frac{1}{2}(1+\gamma_5)U + \frac{1}{2}(1-\gamma_5)U^+,$$

$$U = \exp\{i(\tau_i\pi_i + \eta)\}.$$

The photon A, W, and Z fields can be incorporated to the Lagrangian as well. In terms of the path integral the effective Lagrangian of mesons is obtained by integrating out the quark fields. All the details can be found in Ref. [6]. The effective Lagrangian has been applied to study meson physics [7,8]. Theoretical results agree well with the data.

VMD is a natural result of this theory [6] and the expressions of VMD (ρ, ω, and ϕ) are derived:

$$\frac{1}{2}eg\{-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu\rho_\nu^0 - \partial_\nu\rho_\mu^0) + A^\mu j_\mu^0\},$$

$$\frac{1}{6}eg\{-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu) + A^\mu j_\mu^\omega\},$$

$$-\frac{1}{3\sqrt{2}}eg\{-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu) + A^\mu j_\mu^\phi\}, \tag{4}$$

where g is a universal coupling constant in this theory and has been determined to be 0.39 by fitting $\rho\to ee^+$ [$\Gamma^{th} = 6.53$ keV and $\Gamma^{ex} = (6.77\pm 0.32)$ keV]. The current j_μ^0 is derived from the vertex of $\rho\pi\pi$ [6],

$$\mathcal{L}_{\rho\pi\pi} = \frac{2}{g}f_{\rho\pi\pi}(q^2)\epsilon_{ijk}\rho_i^\mu\pi_j\partial_\mu\pi_k, \tag{5}$$

by substituting

$$\rho_\mu^0 \to \frac{1}{2}egA_\mu$$

into Eq. (5), where

$$f_{\rho\pi\pi}(q^2) = 1 + \frac{q^2}{2\pi^2 f_\pi^2}\left[\left(1-\frac{2c}{g}\right)^2 - 4\pi^2c^2\right] = 1 + 0.243(q^2/\mathrm{GeV}^2) \tag{6}$$

where $c = f_\pi^2/(2gm_\rho^2)$, $f_\pi = 0.186$ GeV, $m_\rho = 0.773$ GeV and q is the momentum of ρ meson. $f_{\rho\pi\pi}(q^2)$ is determined up to the fourth order in derivatives. It is necessary to emphasize that the VMD (4) is derived from this theory, however, the $\rho\pi\pi$ coupling in this theory is no longer a constant. There is a function $f_{\rho\pi\pi}(q^2)$ in the vertex $\mathcal{L}_{\rho\pi\pi}$. $f_{\rho\pi\pi}(q^2)$ is the intrinsic form factor of the pion. The intrinsic form factor is the physical effect of quark loops. There are intrinsic form factors for kaons too (see below). In this paper we apply

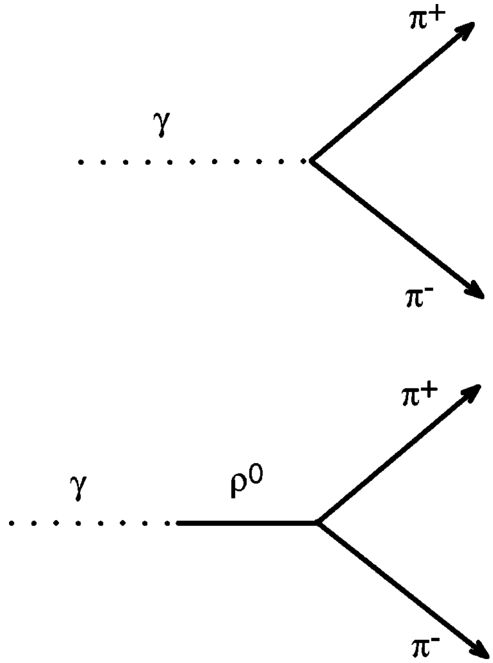

FIG. 1. Feynman diagrams of pion form factor.

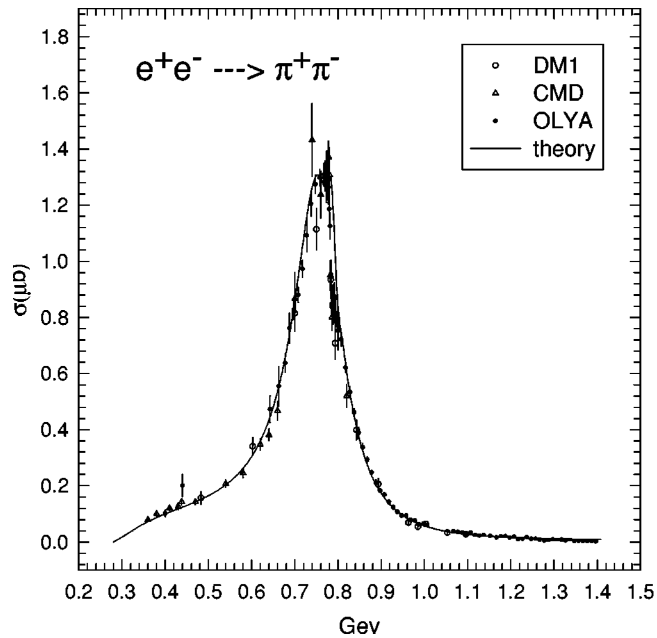

FIG. 2. Cross section of $e^+e^-\to\pi^+\pi^-$ vs invariant mass of $\pi\pi$. Data are from Ref. [12].

VMD (4) in which an intrinsic form factor is included to study form factors of pions and kaons. A detailed comparison between theoretical results of the form factors and experimental data in both timelike and spacelike regions is presented.

VMD (4) shows that the pion form factor is determined by two Feynman diagrams shown in Fig. 1 and obtained from Eqs. (4)–(6)

$$|F_\pi(q^2)|^2=f^2_{\rho\pi\pi}(q^2)\frac{m_\rho^4+q^2\Gamma_\rho^2(q^2)}{(q^2-m_\rho^2)^2+q^2\Gamma_\rho^2(q^2)} \tag{7}$$

in the time-like region, where $\Gamma_\rho(q^2)$ is the decay width of ρ meson.

$$\Gamma_\rho(q^2)=\Gamma_{\rho^0\to\pi^+\pi^-}(q^2)+\Gamma_{\rho^0\to K\bar{K}}(q^2),$$

$$\Gamma_{\rho^0\to\pi^+\pi^-}(q^2)=\frac{f^2_{\rho\pi\pi}(q^2)\sqrt{q^2}}{12\pi g^2}\left(1-\frac{4m^2_{\pi^+}}{q^2}\right)^{3/2}\times\theta(q^2>4m^2_{\pi^+}),$$

$$\Gamma_{\rho^0\to K\bar{K}}(q^2)=\frac{f^2_{\rho\pi\pi}(q^2)\sqrt{q^2}}{48\pi g^2}\left(1-\frac{4m^2_{K^+}}{q^2}\right)^{3/2}\times\theta(q^2>4m^2_{K^+})+\frac{f^2_{\rho\pi\pi}(q^2)\sqrt{q^2}}{48\pi g^2}\times\left(1-\frac{4m^2_{K^0}}{q^2}\right)^{3/2}\theta(q^2>4m^2_{K^0}), \tag{8}$$

when $q^2>4m^2_{K^+}$ the $K\bar{K}$ channel is open. There are other channels, however, in the range of $\sqrt{q^2}<1.4$ GeV the contribution of the other channels is negligible. In the spacelike region the pion form factor is

$$F_\pi(q^2)=\frac{m_\rho^2 f_{\rho\pi\pi}(q^2)}{m_\rho^2-q^2}. \tag{9}$$

From Eqs. (7) and (9) it can be seen that the pion form factor consists of two parts: the intrinsic form factor $f_{\rho\pi\pi}(q^2)$ (6) and the ρ pole. The expression of $f_{\rho\pi\pi}(q^2)$ shows that in the timelike region $f_{\rho\pi\pi}(q^2)$ increases with q^2 and in the spacelike region it decreases with q^2. As mentioned above this behavior is needed to fit the data.

In the timelike region the pion form factor has been measured in $ee^+\to\pi\pi$ and $\tau\to\pi\pi\nu$. The cross section of $ee^+\to\pi^+\pi^-$ is expressed as

$$\sigma=\frac{\pi\alpha^2}{3}\frac{1}{q^2}\left(1-\frac{4m^2_{\pi^+}}{q^2}\right)^{3/2}|F_\pi(q^2)|^2. \tag{10}$$

Taking ρ-ω mixing into account we obtain

$$F_\pi(q^2)=f_{\rho\pi\pi}(q^2)\left(1-\frac{q^2\cos\theta}{q^2-m_\rho^2+i\sqrt{q^2}\Gamma_\rho(q^2)}-\frac{1}{3}\frac{q^2\sin\theta}{q^2-m_\omega^2+i\sqrt{q^2}\Gamma_\omega}\right), \tag{11}$$

where θ is the mixing angle and determined to be 1.74° in Ref. [6]. From Eq. (11), the pion form factor with ρ-ω mixing in the spacelike region is obtained:

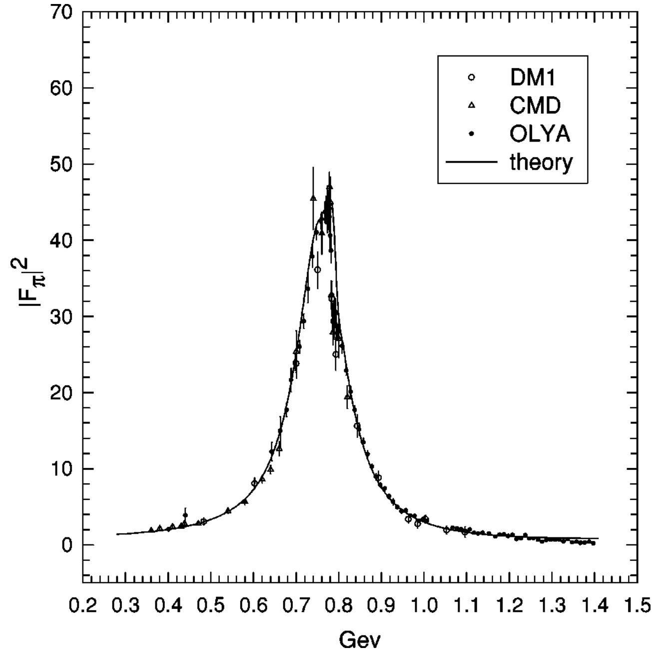

FIG. 3. Pion form factor in a timelike region. Data are from [12].

$$F_\pi(q^2)=f_{\rho\pi\pi}(q^2)\left(1-\frac{q^2\cos\theta}{q^2-m_\rho^2}-\frac{1}{3}\frac{q^2\sin\theta}{q^2-m_\omega^2}\right). \quad (12)$$

The comparison of the cross section (10), the form factor of the pions in both timelike and spacelike regions (11),(12) with data are shown in Figs. 2–5. Up to $\sqrt{q^2}=1.4$ GeV, the channel $\rho\to\pi\pi$ is dominant and the contribution of the $K\bar{K}$ mode is very small. The intrinsic form factor $f_{\rho\pi\pi}(q^2)$ plays a more important role in the range of higher q^2. Using Eq. (8), we obtain $\Gamma_\rho=143.2$ MeV which fits the data well (see Figs. 2 and 3). $f_{\rho\pi\pi}^2(q^2)=1.31$ at $q^2=m_\rho^2$, therefore, the intrinsic form factor makes significant contribution to Γ_ρ. In Ref. [9] a fitting of ρ resonance to $ee^+\to\pi\pi$ has been presented. The radius of charged pion is found from Eq. (12)

$$\langle r^2\rangle_\pi=6\left(\frac{\cos\theta}{m_\rho^2}+\frac{\sin\theta}{3m_\omega^2}\right)+\frac{3}{\pi^2 f_\pi^2}\left[\left(1-\frac{2c}{g}\right)^2-4\pi^2c^2\right]. \quad (13)$$

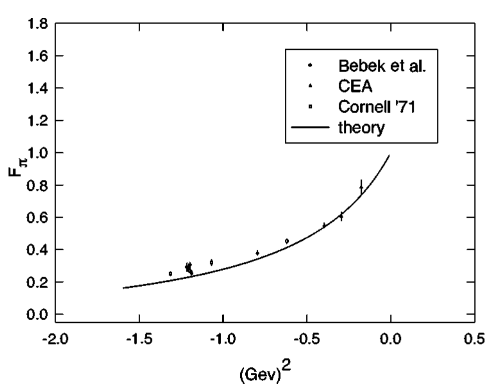

FIG. 4. Pion form factor in a spacelike region. Data are from [13].

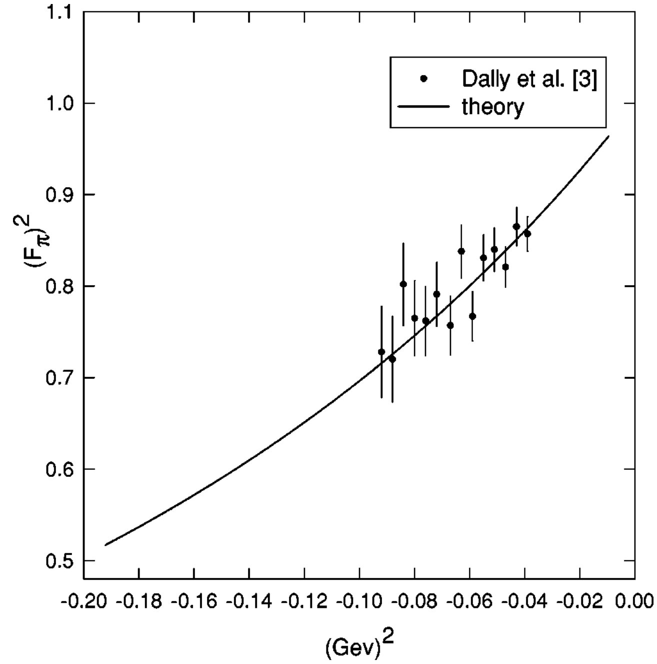

FIG. 5. Pion form factor in a spacelike region.

The numerical result is

$$\langle r^2\rangle_\pi=(0.395+0.057)\ \ \text{fm}^2=0.452\ \ \text{fm}^2, \quad (14)$$

where the first number 0.395 comes from ρ and ω poles and the second is the contribution of the intrinsic form factor (6), which is about 13% of the total value. The experimental data is (0.439 ± 0.03) fm^2 [3].

In terms of the method presented in Ref. [7], the decay rate of $\tau\to\pi\pi\nu$ is derived

$$\frac{d\Gamma}{dq^2}=\frac{G^2}{(2\pi)^3}\frac{\cos^2\theta_C}{48m_\tau^3}(m_\tau^2+2q^2)(m_\tau^2-q^2)^2 \times\left(1-\frac{4m_\pi^2}{q^2}\right)^{3/2}|F_\pi(q^2)|^2, \quad (15)$$

where $F_\pi(q^2)$ is given by Eq. (7). In Ref. [10] it is shown that the form factor $F_\pi(q^2)$ determined from τ decay is consistent with the one obtained from ee^+ annihilation. The branching ratio is calculated to be

$$B_{\tau^-\to\pi^0\pi^-\nu_\tau}=22.3\%. \quad (16)$$

The experimental data are $B_{\tau^-\to\pi^0\pi^-\nu_\tau}=(25.32\pm0.15)\%$ [11].

In the same way the form factors of charged and neutral kaons are studied. All three vector mesons, ρ, ω, and ϕ, contribute to the form factors of kaons. Using the substitutions

$$\phi_\mu\to-\frac{eg}{3\sqrt{2}}A_\mu,\quad \omega_\mu\to\frac{eg}{6}A_\mu, \quad (17)$$

the currents $j_\mu^{\omega,\phi}$ of Eq. (4) are obtained from the vertices [6]

$$\mathcal{L}_{vK\bar{K}}=-\frac{2\sqrt{2}}{g}if_{\rho\pi\pi}(q^2)\phi^\mu(K^+\partial_\mu K^-+K^0\partial_\mu\overline{K^0})$$
$$+\frac{2}{g}if_{\rho\pi\pi}(q^2)\omega^\mu(K^+\partial_\mu K^-+K^0\partial_\mu\overline{K^0})$$
$$+\frac{2}{g}if_{\rho\pi\pi}(q^2)\rho^\mu(K^+\partial_\mu K^--K^0\partial_\mu\overline{K^0}). \qquad (18)$$

The cross sections of $e^+e^-\to K^+K^-$ and $e^+e^-\to K^0\overline{K^0}$ are derived

$$\sigma_{e^+e^-\to K^+K^-}=\frac{\pi\alpha^2}{3}\frac{1}{q^2}\left(1-\frac{4m_{K^+}^2}{q^2}\right)^{3/2}|F_{K^+}|^2,$$
$$\sigma_{e^+e^-\to K^0\bar{K}^0}=\frac{\pi\alpha^2}{3}\frac{1}{q^2}\left(1-\frac{4m_{K^0}^2}{q^2}\right)^{3/2}|F_{K^0}|^2, \qquad (19)$$

where

$$|F_{K^+}(q^2)|^2=f_{\rho\pi\pi}^2(q^2)|A|^2, \qquad (20)$$
$$|F_{K^0}(q^2)|^2=f_{\rho\pi\pi}^2(q^2)|B|^2. \qquad (21)$$

A and B are defined as

$$A=\frac{1}{2}\frac{-m_\rho^2+i\sqrt{q^2}\Gamma_\rho(q^2)}{q^2-m_\rho^2+i\sqrt{q^2}\Gamma_\rho(q^2)}+\frac{1}{6}\frac{-m_\omega^2+i\sqrt{q^2}\Gamma_\omega}{q^2-m_\omega^2+i\sqrt{q^2}\Gamma_\omega}$$
$$+\frac{1}{3}\frac{-m_\phi^2+i\sqrt{q^2}\Gamma_\phi(q^2)}{q^2-m_\phi^2+i\sqrt{q^2}\Gamma_\phi(q^2)}, \qquad (22)$$

$$B=-\frac{1}{2}\frac{-m_\rho^2+i\sqrt{q^2}\Gamma_\rho(q^2)}{q^2-m_\rho^2+i\sqrt{q^2}\Gamma_\rho(q^2)}+\frac{1}{6}\frac{-m_\omega^2+i\sqrt{q^2}\Gamma_\omega}{q^2-m_\omega^2+i\sqrt{q^2}\Gamma_\omega}$$
$$+\frac{1}{3}\frac{-m_\phi^2+i\sqrt{q^2}\Gamma_\phi(q^2)}{q^2-m_\phi^2+i\sqrt{q^2}\Gamma_\phi(q^2)}, \qquad (23)$$

with

$$\Gamma_\phi(q^2)=\Gamma_{\phi\to K^+K^-}(q^2)+\Gamma_{\phi\to K^0\bar{K}^0}(q^2),$$
$$\Gamma_{\phi\to K^+K^-}(q^2)=\frac{\sqrt{q^2}}{24g^2\pi}f_{\rho\pi\pi}^2(q^2)\left(1-\frac{4m_{K^+}^2}{q^2}\right)^{3/2},$$
$$\Gamma_{\phi\to K^0\bar{K}^0}(q^2)=\frac{\sqrt{q^2}}{24g^2\pi}f_{\rho\pi\pi}^2(q^2)\left(1-\frac{4m_{K^0}^2}{q^2}\right)^{3/2}. \qquad (24)$$

The numerical results are

$$\Gamma(\phi\to K^+K^-)=2.14\ \text{MeV},$$
$$\Gamma(\phi\to K^0\overline{K^0})=1.4\ \text{MeV},$$

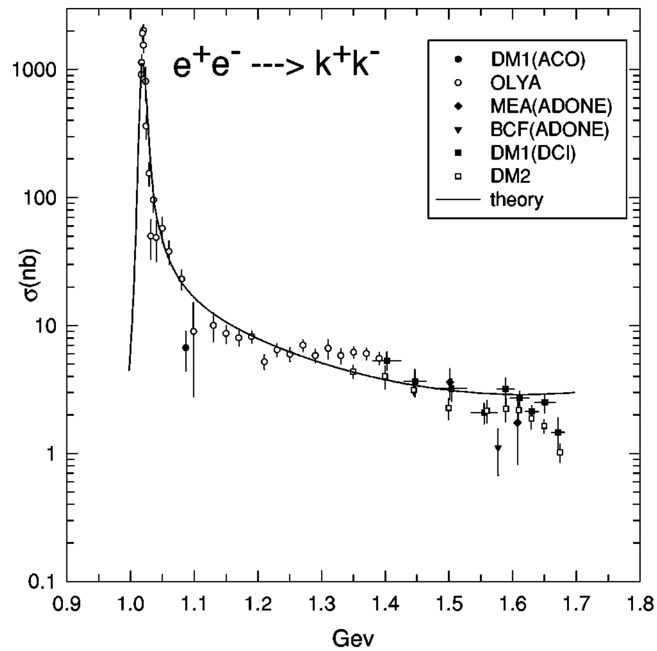

FIG. 6. Cross section of $e^+e^-\to K^+K^-$ vs invariant mass of KK. Data are from [14,15].

$$\Gamma(\phi\to K^+K^-)_{exp}=2.18(1\pm0.028)\ \text{MeV},$$
$$\Gamma(\phi\to K^0\overline{K^0})_{exp}=1.51(1\pm0.029)\ \text{MeV}. \qquad (25)$$

$f_{\rho\pi\pi}^2(q^2)=1.57$ at $q^2=m_\phi^2$. The contribution of the intrinsic forn factor is significant. Theoretical results of Γ_ϕ agree with the data very well. Γ_ω has been calculated to be 7.7 MeV [7] which is consistent with the data 7.49 (1 ± 0.02) MeV.

The cross sections of $ee^+\to K\bar{K}$ and the form factors of kaons in timelike region are shown in Figs. 6–9.

The form factors of kaons in spacelike region are obtained from Eqs. (20)–(23)

$$F_{K^+}(q^2)=f_{\rho\pi\pi}(q^2)\left(\frac{1}{2}\frac{m_\rho^2}{m_\rho^2-q^2}+\frac{1}{6}\frac{m_\omega^2}{m_\omega^2-q^2}+\frac{1}{3}\frac{m_\phi^2}{m_\phi^2-q^2}\right),$$
$$F_{K^0}(q^2)=f_{\rho\pi\pi}(q^2)\left(-\frac{1}{2}\frac{m_\rho^2}{m_\rho^2-q^2}+\frac{1}{6}\frac{m_\omega^2}{m_\omega^2-q^2}+\frac{1}{3}\frac{m_\phi^2}{m_\phi^2-q^2}\right). \qquad (26)$$

The form factor, $F_{K^+}(q^2)$, is shown in Fig. 10. The radius of the charged kaon is derived from Eq. (26)

$$\langle r^2\rangle_{K^+}=\frac{3}{\pi^2f_\pi^2}\left[\left(1-\frac{2c}{g}\right)^2-4\pi^2c^2\right]+\left(\frac{3}{m_\rho^2}+\frac{2}{m_\phi^2}+\frac{1}{m_\omega^2}\right). \qquad (27)$$

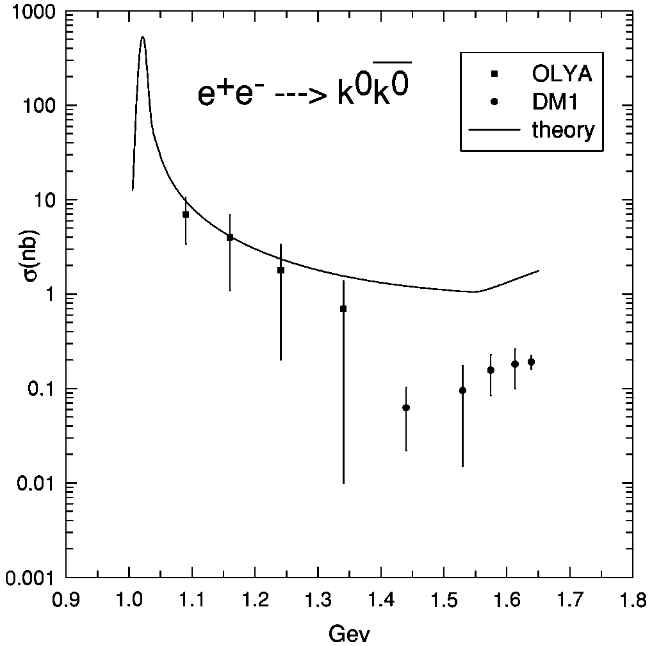

FIG. 7. Cross section of $e^+e^-\to K^0\overline{K^0}$ vs invariant mass of KK. Data are from [14,15].

The numerical value is

$$\langle r^2\rangle_{K^+}=(0.05+0.33)\ \ \mathrm{fm}^2=0.38\ \ \mathrm{fm}^2. \tag{28}$$

The first number comes from the intrinsic form factor. The experimental data is (0.34 ± 0.05) fm^2 [16]. The radius of the neutral kaon is obtained from Eq. (26)

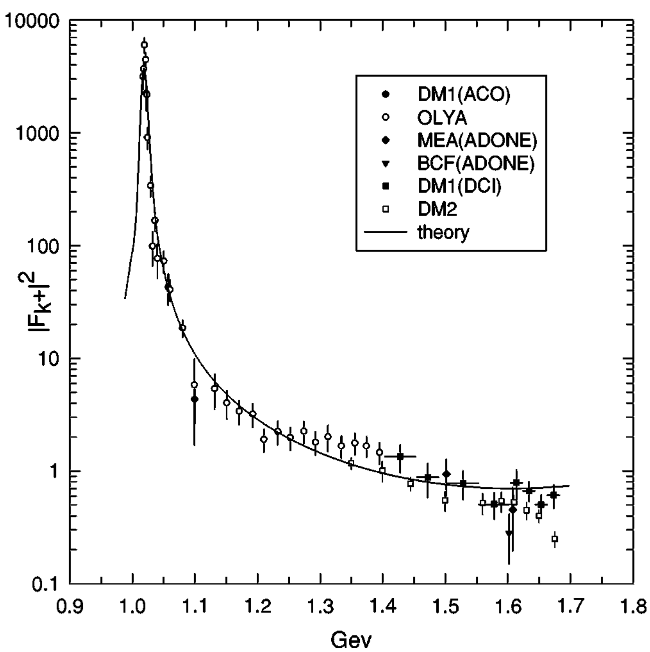

FIG. 8. Charged kaon form factor in a timelike region. Data are from [14–16].

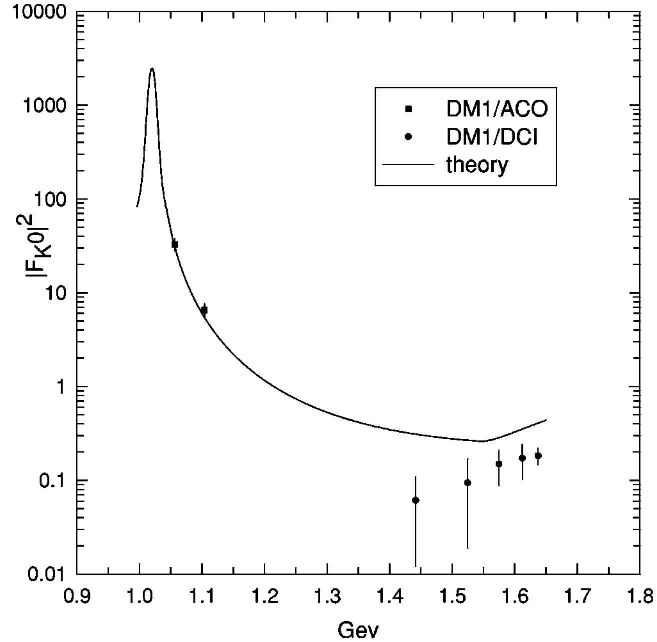

FIG. 9. Neutral kaon form factor in a timelike region. Data are from [14–16].

$$\begin{aligned}\langle r^2\rangle_{K^0}&=-6\frac{\partial F_{K^0}(q^2)}{\partial q^2}\Big|_{q^2=0}\\&=-\frac{2}{m_{\phi^2}}-\frac{1}{m_\omega^2}+\frac{3}{m_\rho^2}\\&=0.057\ \ \mathrm{fm}^2.\end{aligned} \tag{29}$$

Only the coupling of $\rho K\overline{K}$ in Eq. (18) contributes to $\tau^-\to K^0K^-\nu$. The decay rate is found to be

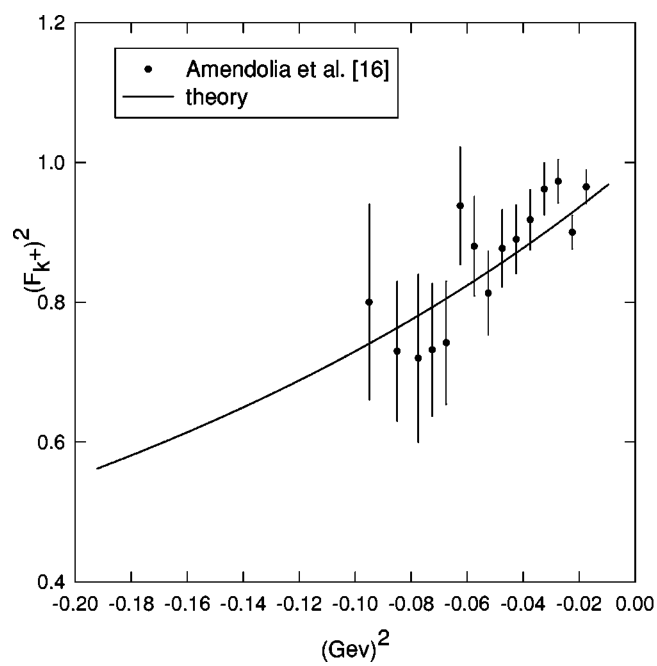

FIG. 10. Charged kaon form factor in a spacelike region.

$$\frac{d\Gamma}{dq^2}=\frac{G^2}{(2\pi)^3}\frac{\cos^2\theta_C}{96m_\tau^3}(m_\tau^2+2q^2)(m_\tau^2-q^2)^2 \times\left(1-\frac{4m_K^2}{q^2}\right)^{3/2}|F_\pi(q^2)|^2, \quad (30)$$

where $F_\pi(q^2)$ is given by Eq. (7). The theory predicts that the decay rate of this process is determined by the pion form factor in the chiral limit. So far there is no measurement on this distribution. The branching ratio of this decay mode is calculated to be

$$B_{\tau^-\to K^0K^-\nu_\tau}=1.78\times10^{-3}. \quad (31)$$

The experimental data is $(1.59\pm0.24)\times10^{-3}$ [11]. Theory agrees with the data within the error bar.

In summary, VMD derived from the chiral theory [6] has been applied to study the form factors of pions and kaons. An intrinsic form factor has been predicted by this chiral theory [6]. It has been found that this intrinsic form factor makes significant contribution to the form factors of pions and kaons, the decay widths of ρ and ϕ mesons, and the decays $\tau\to\pi\pi\nu$ and $\tau\to K\bar{K}\nu$. The theory agrees well with the data. It is necessary to point out that there is no new parameter in this study. In this theory derivative expansion is used and all the calculations are done up to the fourth order in derivatives. The fitting shows that the pion form factor is in agreement with data up to $q\sim1.4$ GeV in the timelike region (Figs. 2, 3). In the range of higher q^2 the contribution of higher order derivatives should be taken into account. We will provide the study in the near future.

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[1] J.J. Sakurai, *Currents and Mesons* (University of Chicago Press, Chicago, 1969).
[2] G.J. Gounaris and J.J. Sakurai, Phys. Rev. Lett. **21**, 244 (1968).
[3] E.B. Dally *et al.*, Phys. Rev. Lett. **48**, 375 (1982).
[4] E. Amaldi, S. Fubini, and G. Furlan, *Pion-Electroproduction* (Springer-Verlag, Berlin, 1979).
[5] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[6] B.A. Li, Phys. Rev. D **52**, 5165 (1995); **52**, 5184 (1995).
[7] B.A. Li, Phys. Rev. D **55**, 1436 (1997); **55**, 1425 (1997). In the first paper the calculation of $\tau\to K\bar{K}\nu$ has an error. The correct result is provided in this paper (31).
[8] D.N. Gao, B.A. Li, and M.L. Yan, Phys. Rev. D **56**, 4115 (1997); B.A. Li, D.N. Gao, and M.L. Yan, *ibid.* **58**, 094031 (1998).
[9] M. Benayoun *et al.*, Z. Phys. C **58**, 31 (1993).
[10] J. Urheim, Nucl. Phys. B (Proc. Suppl.) **55C**, 359 (1997).
[11] Particle Data Group, C. Caso *et al.*, Eur. Phys. J. C **3**, 1 (1998).
[12] A. Quenzer *et al.*, Phys. Lett. **76B**, 512 (1978); L.M. Barkov *et al.*, Nucl. Phys. **B256**, 365 (1985).
[13] C.J. Bebek *et al.*, Phys. Rev. D **13**, 25 (1976).
[14] B. Delcourt *et al.*, Phys. Lett. **99B**, 257 (1981); DM2 Collaboration, D. Bisello *et al.*, Z. Phys. C **39**, 13 (1988).
[15] N.N. Achasov and A.A. Kozhevnikov, Phys. Rev. D **57**, 4334 (1998); F. Mane *et al.*, Phys. Lett. **99B**, 261 (1981).
[16] S.R. Amendolia *et al.*, Phys. Lett. B **178**, 435 (1986).